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The Gibbs Phenomenon for Fourier Interpolation

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The Fourier interpolation polynomials for periodic functions with an isolated jump discontinuity exhibit for growing order a Gibbs phenomenon. The over- and undershots differ, however, from the ones appearing for the partial sums of the Fourier series and depend on the coincidence of the jump with interpolation nodes.

1. THE PROBLEM

Let f be a real-valued periodic function on $\mathbb{R}$ with period 2π, of bounded variation on $[-\pi, \pi]$, and with an isolated jump discontinuity of size $f^+(\xi) - f^-(\xi) = 2s$ in ξ. Let

$$s_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

be the nth partial sum of the Fourier series of f. Then

$$\lim_{n\to\infty} s_n\left(\xi + \frac{\pi}{n}\right) - \frac{f^+(\xi) + f^-(\xi)}{2} = s \cdot S_0,$$

where $S_0 = (2/\pi)\int_0^\pi (\sin t/t)\,dt = 1{,}1789\ldots$ [8, II.9; 4]. In addition, this value S_0 has the character of a supremum in the sense

$$S_0 = \lim_{\varepsilon \searrow 0} \limsup_{n\to\infty} \max_{0<\theta<\varepsilon} \left| s_n(\xi+\theta) - \frac{f^+(\xi) + f^-(\xi)}{2} \right| \Big/ |s|.$$

In view of the importance gained by the fast Fourier transform it seems a legitimate question to ask whether such a phenomenon also appears in discrete Fourier interpolation. Here (following [6, 4.1; 9, X.3])

$$s_n^*(x) = \frac{a_0^*}{2} + \sum_{k=1}^{n-1} (a_k^* \cos kx + b_k^* \sin kx) + \frac{a_n^*}{2} \cos nx$$

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is a trigonometric polynomial of order n which for the nodes

$$x_j = x_{n,j} = j\,\frac{\pi}{n} \qquad (1 \leqslant j \leqslant 2n) \tag{1}$$

satisfies

$$s_n^*(x_j) = f(x_j) \qquad (1 \leqslant j \leqslant 2n),$$

and s_n^* is determined by these requirements uniquely up to an additive term $b^* \sin nx$ which vanishes automatically at each node x_j.

At a first glance a positive answer might seem trivial since for $n \to \infty$ one has

$$a_k^* = a_{n,k}^* = \frac{1}{n} \sum_{j=1}^{2n} f(x_j) \cos kx_j \to a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos kt\, dt,$$

$$b_k^* = b_{n,k}^* = \frac{1}{n} \sum_{j=1}^{2n} f(x_j) \sin kx_j \to b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin kt\, dt;$$

so one would expect s_n^* to behave like s_n. On the second thought, however, one realizes that this impression is based not on an approximation of s_n by s_n^* but on an approximation of s_m for fixed m by the mth order part of s_n^* for suficiently large n. A reasoning which wants to rely on a uniform approximation of s_n by s_n^* would require estimates such as

$$|f(x) \cos kx - f(x_j) \cos kx_j| < \frac{\varepsilon}{n} \quad \text{for } x_{j-1} \leqslant x \leqslant x_j,\ 1 \leqslant j \leqslant 2n,\ 1 \leqslant k \leqslant n$$

which are invalid for points near a jump and not guaranteed elswhere wthout further hypotheses.

Reformulating a result of de la Vallée-Poussin [7], Zygmund [9, X. Problem 10, 11] states that under the suppositions

$$x_m < \xi < x_{m+1},$$

$$\theta_n = n(\xi - x_m)/\pi,$$

$$\psi(\theta) = \frac{\sin \pi\theta}{\pi} \left(\frac{1}{\theta} - \frac{1}{\theta+1} + \frac{1}{\theta+2} - \cdots \right),$$

one has

$$\lim_{n \to \infty} \{ s_n^*(\xi) - [\psi(\theta_n) f^-(\xi) + \psi(1-\theta_n) f^+(\xi)] \} = 0.$$

This does not immediately give an answer to the question at hand. Neither does the result of Foster and Richards [2] who consider least squares approximation of f by continuous functions which are linear on the intervals with endpoints x_j $(1 \leqslant j \leqslant 2n)$. There the Gibbs phenomenon may be established by computing the ordinates of the approximating function at the nodes. Although Fourier interpolation may be considered as a least squares approximation problem [6, 4.6; 9, X.3], the ordinates of the approximating trigonometric polynomials distant from the nodes (in which they coincide with the values of f) do not roll out as readily from the calculation of the approximating function.

Runck [5] studies the analogous problem for equidistant Lagrange interpolation if the jump occurs in a node. Bojanic and Cheng [1] consider Hermite–Fejér interpolation of f on $[-1, 1]$ by a polynomial $H_n(f, \cdot)$. They show

$$\limsup_{n \to \infty} H_n(f, \xi) - \frac{f^+(\xi) + f^-(\xi)}{2} = |s| \cdot \beta(\xi)$$

(and a similar formula for lim inf) where $\xi = \cos \alpha\pi$ and where $\beta(\xi)$ depends on the fact whether α is irrational (in which case $\beta(\xi) = 1$) or not. This result again does not furnish information on the behaviour of H_n in neighbourhoods of ξ which decrease as $n \to \infty$. Still, the theorem offered in the next section reflects the dependence on the arithmetical properties of the position of ξ as exhibited above.

2. An Answer

As a partial answer to the above-mentioned question the following theorem is offered.

Theorem 1. *Let f be a real-valued periodic function on $\mathbb{R}$ with period 2π, of bonded variation on $[-\pi, \pi]$, and with an isolated jump discontinuity in $\xi \in [-\pi, \pi]$ satisfying*

$$f^+(\xi) + f^-(\xi) = 2f(\xi), \tag{2}$$

$$f^+(\xi) - f^-(\xi) = 2s \tag{3}$$

Given n let m be determined by

$$m\frac{\pi}{n} = x_m \leqslant \xi < x_{m+1} = (m+1)\frac{\pi}{n}. \tag{4}$$

Then for $n \to \infty$ one has

$$s_n^*\left(x_{m+1}+\frac{\pi}{2n}\right)-\frac{f^+(\xi)+f^-(\xi)}{2}=s\cdot S(\xi-x_m)+o(1),$$

where

$$\begin{aligned} S(0)&=\frac{10}{3\pi}=1{,}061\ldots,\\ S(t)&=\frac{4}{\pi}=1{,}273\ldots \qquad \textit{for} \quad t>0. \end{aligned} \tag{5}$$

Proof. For any $n\in\mathbb{N}$ the interpolating trigonometric polynomial s_n^* may be written as

$$s_n^*(x)=\frac{1}{n}\sum_{j=1}^{2n} f(x_j)\,D_n^*(x-x_j),$$

where the modified Dirichlet kernel D_n^* is given by

$$\begin{aligned} D_n^*(u)&=\frac{1}{2}+\sum_{k=1}^{n-1}\cos ku+\frac{1}{2}\cos nu\\ &=\frac{1}{2}\sin nu\cot\frac{u}{2} \end{aligned}$$

[9, X.3]. D_n^* is an even function satisfying

$$\begin{aligned} D_n^*(0)&=n,\\ D_n^*(\pi)&=0,\\ D_n^{*\prime}(0)&=D_n^{*\prime}(\pi)=0. \end{aligned}$$

By the linearity of the interpolation and by the theorem on uniform convergence on closed intervals of continuity [7, X.5.4] it suffices to consider the function f defined by

$$f(x)=\begin{cases} -1 & \text{for} \quad \xi-\pi<x<\xi\\ 0 & \text{for} \quad x=\xi, \quad x=\xi+\pi\\ 1 & \text{for} \quad \xi<x<\xi+\pi \end{cases} \tag{6}$$

with jumps of size $2s=2$ at ξ and $\xi+\pi$. Further on we therefore consider the polynomial

$$s_n^*(x)=\frac{1}{n}\sum_{j=m+1}^{m+n^*}\left[D_n^*(x-x_j)-D_n^*(x-x_{j+n})\right], \tag{7}$$

where

$$n^* = \begin{cases} n & \text{if} \quad x_m < \xi, \\ n-1 & \text{if} \quad x_m = \xi. \end{cases} \tag{8}$$

Note that the functions $s_n^*(x_m + x)$ resp. $s_n^*(x_m + \pi/2 + x)$ are odd resp. even in x if $x_m = \xi$, while in the case $x_m < \xi$ this applies to the functions $s_n^*(x_m + \pi/2n + x)$ resp. $s_n^*(x_m + \pi/2n + \pi/2 + x)$. Applying elementary trigonometric identities one obtains

$$D_n^*(u) - D_n^*(u - \pi) = \begin{cases} \dfrac{\sin nu}{\sin u} & \text{if} \quad n \text{ is even} \\ \dfrac{\sin nu}{\sin u} \cos u & \text{if} \quad n \text{ is odd} \end{cases} \tag{9}$$

and

$$\sin n(x - x_j) = (-1)^j \sin nx.$$

Suppose first that n is even. Then

$$s_n^*(x) = \frac{(-1)^m}{n} \sum_{j=1}^{n^*} (-1)^j \frac{\sin nx}{\sin(x - x_{m+j})}$$

$$s_n^*\left(x_{m+1} + \frac{\pi}{2n}\right) = \frac{(-1)^m}{n} \sum_{j=1}^{n^*} (-1)^{j+1} \frac{\sin((2m+3)/2)\pi)}{\sin(((j-1)/n)\pi - \pi/2n)}$$

$$= \frac{1}{n} \sum_{j=1}^{n^*} \frac{(-1)^j}{\sin(((2j-3)/2n)\pi)}. \tag{10}$$

Note that $1/\sin x$ is symmetric about $x = \pi/2$. Therefore the last sum reduces to $2/\sin(\pi/2n)$ if $n^* = n$ ($x_m < \xi$) and to $2/\sin(\pi/2n) - 1/\sin(3\pi/2n)$ if $n^* = n - 1$ ($x_m = \xi$). The assertion of the theorem now follows from

$$\lim_{n \to \infty} \frac{2}{n \sin(\pi/2n)} = \frac{4}{\pi},$$

$$\lim_{n \to \infty} \left(\frac{2}{n \sin(\pi/2n)} - \frac{1}{n \sin(3\pi/2n)}\right) = \frac{1}{\pi}\left(4 - \frac{2}{3}\right).$$

If n is odd then by (7) and (9)

$$s_n^*\left(x_{m+1} + \frac{\pi}{2n}\right) = \frac{1}{n} \sum_{j=1}^{n^*} (-1)^j \cot\left(\frac{2j-3}{2n}\pi\right).$$

The fact that $\cot(x - \pi/2)$ is odd, together with the alternating signs, again reduces the last sum to $2\cot(\pi/2n)$ resp. $2\cot(\pi/2n) - \cot(3\pi/2n)$ with the same limits as above. ∎

By the same token and under the assumptions (2), (3), (4) one obtains

COROLLARY. *Let*

$$S_k = 2 \cdot \left(1 - \frac{1}{3} + \cdots + \frac{(-1)^k}{2k+1}\right) = 2 \sum_{l=0}^{k} \frac{(-1)^l}{2l+1} \qquad (k = 0, 1, \ldots). \tag{11}$$

Then for $n \to \infty$ one has

$$s_n^*\left(x_{m+1} + (2k+1)\frac{\pi}{2n}\right) - \frac{f^+(\xi) + f^-(\xi)}{2}$$

$$= s \cdot \begin{cases} \dfrac{2}{\pi} S_k + o(1) & \text{if } x_m < \xi, \\ \dfrac{2}{\pi}\left(S_k + \dfrac{(-1)^{k+1}}{2k+3}\right) + o(1) & \text{if } x_m = \xi. \end{cases} \tag{12}$$

This agrees with (5) and with the fact that

$$\lim_{k \to \infty} S_k = 2 \sum_{l=0}^{\infty} \frac{(-1)^l}{2l+1} = \frac{\pi}{2}.$$

In particular for $k = 1$ we get an undershot of s times $8/3\pi = 0{,}848\ldots$ resp. s times $46/15\pi = 0{,}976\ldots$.

The requirement (2), which we also presuppose in the sequel, is motivated by the fact that for the partial sum s_n of the Fourier series for f one has

$$\lim_{n \to \infty} s_n(\xi) = \frac{f^+(\xi) + f^-(\xi)}{2}.$$

This requirement is irrelevant in the case $x_m < \xi$. If, however, $x_m = \xi$ and

$$c = \left[f(\xi) - \frac{f^+(\xi) + f^-(\xi)}{2}\right] \Big/ s \neq 0,$$

then one necessarily has

$$s_n^*(\xi) = f(\xi) \neq \frac{f^+(\xi) + f^-(\xi)}{2}.$$

A calculation as in the proof of the theorem shows that for $k = 0, 1, 2, \ldots$ and $n \to \infty$ one then has

$$s_n^*\left(x_{m+1} + (2k+1)\frac{\pi}{2n}\right) - \frac{f^+(\xi) + f^-(\xi)}{2}$$
$$= s \cdot \frac{2}{\pi}\left[S_k + \frac{(-1)^{k+1}}{2k+3}(1+c)\right] + o(1), \tag{13}$$

$$s_n^*\left(x_{m+1} \pm \frac{\pi}{2n}\right) - \frac{f^+(\xi) + f^-(\xi)}{2} = s \cdot \frac{2}{\pi}(c \pm 1) + o(1). \tag{14}$$

If $c = -1$ ($f(\xi) = f^-(\xi)$) then (13) agrees with the assertion for $x_m < \xi$. Formulas (13) and (14) point out that, as soon as the jump coincides with an interpolation node, left or right continuity of f introduce an asymmetry in the location of the Gibbs phenomenon. Moreover, by these formulas one may expect that, in case of a jump at an interpolation node, putting $c = 0$ introduces a notable and even maximal reduction of the Gibbs phenomenon. This again motivates the natural assumption (2) and the special attention to the case $\xi = x_m$ in the rest of this paper.

3. The Situation of Local Extrema

The theorem is incomplete insofar as it does not furnish information about the behaviour of s_n^* between the nodes x_j (where s_n^* agrees with f) and the points $x_j + \pi/2n$. In particular, the values indicated in the theorem do not provide the maximal overshot. In order to obtain further relevant information we compute the derivatives of s_n^* in the just mentioned points. In order to simplify the notation we write

$$\bar{x}_j = \frac{x_j + x_{j+1}}{2}. \tag{15}$$

Note that

$$D_n^{*\prime}(u) = -\frac{1}{2}\left(2\sum_{k=1}^{n-1} k \sin ku + n \sin nu\right),$$
$$D_n^{*\prime\prime}(u) = -\frac{1}{2}\left(2\sum_{k=1}^{n-1} k^2 \cos ku + n^2 \cos nu\right).$$

By induction on n we get

$$D_n^{*\prime\prime}(0) = -\frac{1}{2}\left(2\sum_{k=1}^{n-1} k^2 + n^2\right) = -\frac{n}{6}(2n^2+1),$$
$$D_n^{*\prime\prime}(\pi) = -\frac{1}{2}\left(2\sum_{k=1}^{n-1} (-1)^k k^2 + (-1)^n n^2\right) = (-1)^{n+1} \cdot \frac{n}{2}.$$

This shows that if $x = x_j$, then the corresponding term in the sum for

$$s_n{}^{*\prime\prime}(x) = \frac{1}{n} \sum_{j=m+1}^{m+n^*} [D_n^{*\prime\prime}(x - x_j) - D_n^{*\prime\prime}(x - x_j - \pi)]$$

has to be replaced by $-(n/3)(n^2-1)$ if n is even and by $-(n/3)(n^2+2)$ if n is odd, while in the sum for $s_n^{*\prime}(x)$ it has to be replaced by 0.

In the following we consider in detail the case of an even n. The case of an odd n leads to the same results and will briefly be treated afterwards.

We now have

$$s_n^{*\prime}(x) = \frac{1}{n} \sum_{j=1}^{n^*} (-1)^{m+j+1} \frac{n \cos nx \sin(x_{m+j} - x) + \sin nx \cos(x_{m+j} - x)}{\sin^2(x_{m+j} - x)}$$

$$s_n^{*\prime\prime}(x) = \frac{1}{n} \sum_{j=1}^{n^*} (-1)^{m+j+1}$$
$$\times \frac{n \cos nx \sin 2(x_{m+j} - x) + \sin nx[2 - (n^2+1) \sin^2(x_{m+j} - x)]}{\sin^3(x_{m+j} - x)}$$

with eventual replacements as just stated. This gives

$$s_n^{*\prime}(x_{m+k}) = \sum_{\substack{j=1 \\ j \neq k}}^{n^*} \frac{(-1)^{k+j}}{\sin(x_{k-j})}, \tag{16}$$

$$s_n^{*\prime}\left(x_{m+k} + \frac{\pi}{2n}\right) = \frac{1}{n} \sum_{j=1}^{n^*} (-1)^{k+j+1} \frac{\cos(\bar{x}_{k-j})}{\sin^2(\bar{x}_{k-j})}, \tag{17}$$

$$s_n^{*\prime\prime}(x_{m+k}) = 2 \cdot \sum_{\substack{j=1 \\ j \neq k}}^{n^*} (-1)^{k+j+1} \frac{\cos(x_{k-j})}{\sin^2(x_{k-j})} - \frac{n^2-1}{3}, \tag{18}$$

$$s_n^{*\prime\prime}\left(x_{m+k} + \frac{\pi}{2n}\right) = \frac{1}{n} \sum_{j=1}^{n^*} (-1)^{k+j} \frac{2 - (n^2+1) \sin^2(\bar{x}_{k-j})}{\sin^3(\bar{x}_{k-j})}. \tag{19}$$

The followig lemma will be used to estimate the alternating sums above.

Lemma 1. *Suppose the non-negative non-increasing sequence $\{a_j\}_{j=0}^n$ satisfies*

$$a_0 - a_1 \geqslant a_1 - a_2 \geqslant \cdots \geqslant a_{n-1} - a_n \tag{20}$$

and let

$$s = \sum_{j=0}^{n} (-1)^j a_j.$$

If n is even then

$$a_0 - \frac{a_1 - a_n}{2} \geqslant s \geqslant \frac{a_0}{2}. \tag{21}$$

If n is odd then

$$a_0 - \frac{a_1 + a_n}{2} \geqslant s \geqslant \frac{a_0 - a_n}{2}. \tag{22}$$

In particular, if $a_n = 0$ then (whether n is even or odd)

$$a_0 - \frac{a_1}{2} \geqslant s \geqslant \frac{a_0}{2}. \tag{23}$$

Proof. If n is even then

$$\begin{aligned} s &= (a_0 - a_1) + (a_2 - a_3) + \cdots + (a_{n-2} - a_{n-1}) + a_n \\ &\geqslant (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{n-1} - a_n) = a_0 - s \\ &\geqslant (a_2 - a_3) + (a_4 - a_5) + \cdots + (a_n - a_n) = s - a_n - a_0 + a_1. \end{aligned}$$

If n is odd then

$$\begin{aligned} s &= (a_0 - a_1) + (a_2 - a_3) + \cdots + (a_{n-3} - a_{n-2}) + (a_{n-1} - a_n) \\ &\geqslant (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{n-2} - a_{n-1}) = a_0 - a_n - s \\ &\geqslant (a_2 - a_3) + (a_4 - a_5) + \cdots + (a_{n-1} - a_n) = s - a_0 + a_1. \quad \blacksquare \end{aligned}$$

Recall that a function g is called convex in $[\alpha, \beta]$ if

$$\theta g(\alpha) + (1-\theta) g(\beta) \geqslant g(\theta\alpha + (1-\theta)\beta) \qquad \text{for} \quad \theta \in [0, 1].$$

For a twice differentiable function g this is equivalent with $g'' \geqslant 0$ in $[\alpha, \beta]$ (for a concave function change $\geqslant$ to $\leqslant$). Note that the conditions of the lemma are satisfied if $a_j = g(x_0 + jh)$ $(0 \leqslant j \leqslant n)$ where $h > 0$ and where g is non-negative, non-increasing, and convex in $[x_0, x_0 + nh]$. Also two consecutive inequality signs in (20) imply inequality signs in the resulting estimates of the lemma.

We now want to estimate (16). In view of the symmetry of $\sin x$ with respect to $x = \pi/2$ we restrict our interest to $1 \leqslant k \leqslant n/2$. Since $\sin x$ is an odd function the first $2k - 2$ terms of the sum cancel each other pairwise while the remaining pairs of terms with $(j - k)$ symmetric with respect to $n/2$ are equal in value.

In the case $n^* = n$ $(x_m < \xi)$ we therefore get

$$s_n^{*\prime}(x_{m+k}) = 2 \sum_{j=2k}^{n/2+k} \frac{(-1)^{k+j+1}}{\sin(x_{j-k})} - (-1)^{n/2+1}.$$

Obviously the sign of $s_n^{*\prime}(x_{m+k})$ is determined by the first sign in the sum which is $(-1)^{k+1}$. Therefore we have

$$(-1)^{k+1}\, s_n^{*\prime}(x_{m+k}) > 0. \tag{24}$$

The function $1/\sin x$ is positive, decreasing, and convex in $]0, \pi/2]$. Applying the lemma and letting $n \to \infty$ we get in case $n/2 - k$ is even

$$(-1)^{k+1}\, s_n^{*\prime}(x_{m+k}) > \frac{1}{\sin x_k} - 1 = \frac{n}{k\pi} + o(n), \tag{25}$$

$$(-1)^{k+1}\, s_n^{*\prime}(x_{m+k}) < \frac{2}{\sin x_k} - \frac{1}{\sin x_{k+1}} = \frac{n}{\pi}\left(\frac{2}{k} - \frac{1}{k+1}\right) + o(n) \tag{26}$$

and in case $n/2 - k$ is odd

$$(-1)^{k+1}\, s_n^{*\prime}(x_{m+k}) > \frac{1}{\sin x_k} = \frac{n}{k\pi} + o(n), \tag{25'}$$

$$(-1)^{k+1}\, s_n^{*\prime}(x_{m+k}) < \frac{2}{\sin x_k} - \frac{1}{\sin x_{k+1}} = \frac{n}{\pi}\left(\frac{2}{k} - \frac{1}{k+1}\right) + o(n). \tag{26'}$$

Suppose now $n^* = n - 1$ $(x_m = \xi)$. Then

$$s_n^{*\prime}(x_{m+n/2}) = 0 \tag{27}$$

while for $k < n/2$ we have

$$s_n^{*\prime}(x_{m+k}) = 2 \sum_{j=k+1}^{n/2-1} \frac{(-1)^{j+1}}{\sin(x_j)} + \frac{(-1)^{k+1}}{\sin(x_k)} + (-1)^{n/2+1}.$$

Applying the appropriate estimates of the lemma we get if $n/2 - k$ is odd

$$\begin{aligned}(-1)^{k+1}\, s_n^{*\prime}(x_{m+k}) &> \frac{1}{\sin x_k} - \frac{2}{\sin x_{k+1}} + \frac{1}{\sin x_{k+2}} + \frac{1}{\sin x_{n/2-1}} + (-1)^{n/2+k} \\ &= \frac{n}{\pi}\left(\frac{1}{k} - \frac{2}{k+1} + \frac{1}{(k+2)}\right) + o(n) \\ &= \frac{2n}{\pi \cdot k(k+1)(k+2)} + o(n),\end{aligned} \tag{28}$$

$$(-1)^{k+1} s_n^{*\prime}(x_{m+k}) < \frac{1}{\sin x_k} - \frac{1}{\sin x_{k+1}} + \frac{1}{\sin x_{n/2-1}} + (-1)^{n/2+k}$$

$$= \frac{n}{\pi}\left(\frac{1}{k} - \frac{1}{k+1}\right) + o(n) \tag{29}$$

and if $n/2 - k$ is even

$$(-1)^{k+1} s_n^{*\prime}(x_{m+k}) > \frac{1}{\sin x_k} - \frac{2}{\sin x_{k+1}} + \frac{1}{\sin x_{k+2}} - \frac{1}{\sin x_{n/2-1}} + (-1)^{n/2+k}$$

$$= \frac{n}{\pi}\left(\frac{1}{k} - \frac{2}{k+1} + \frac{1}{(k+2)}\right) + o(n), \tag{28'}$$

$$(-1)^{k+1} s_n^{*\prime}(x_{m+k}) < \frac{1}{\sin x_k} - \frac{1}{\sin x_{k+1}} + (-1)^{n/2+k}$$

$$= \frac{n}{\pi}\left(\frac{1}{k} - \frac{1}{k+1}\right) + o(n). \tag{29'}$$

Next we are going to estimate (17), again for $1 \leqslant k \leqslant n/2$. Now since $\cos x/\sin^2 x$ is an even function the first $2k$ terms of the sum cancel each other pairwise while the remaining pairs of terms with $(j-k)$ symmetric with respect to $(n+1)/2$ again are equal in value.

If $n^* = n$ $(x_m < \xi)$ then we obtain

$$s_n^{*\prime}(\bar{x}_{m+n/2}) = 0 \tag{30}$$

while for $k < n/2$ we have

$$s_n^{*\prime}(\bar{x}_{m+k}) = \frac{2}{n} \sum_{j=2k+1}^{n/2+k} (-1)^{k+j+1} \frac{\cos(\bar{x}_{k-j})}{\sin^2(\bar{x}_{k-j})}.$$

Here the sign of $s_n^{*\prime}(\bar{x}_{m+k})$ is $(-1)^k$, so we have

$$(-1)^k s_n^{*\prime}(\bar{x}_{m+k}) > 0. \tag{31}$$

The function $\cos x/\sin^2 x$ is non-negative, decreasing, and convex in $]0, \pi/2]$. Note that the convexity requirement (20) of the sequence $\{\cos \bar{x}_{k-j}/\sin^2 \bar{x}_{k-j}\}_{j=2k+1}^{n/2+k}$ is not affected by adjoining a last term 0. Applying the lemma we get as $n \to \infty$

$$(-1)^k s_n^{*\prime}(\bar{x}_{m+k}) > \frac{\cos(\bar{x}_k)}{n \sin^2(\bar{x}_k)} = \frac{4n}{(2k+1)^2 \pi^2} + o(n), \tag{32}$$

$$(-1)^k s_n^{*\prime}(\bar{x}_{m+k}) < \frac{2}{n}\left[\frac{\cos \bar{x}_k}{\sin^2 \bar{x}_k} - \frac{\cos \bar{x}_{k+1}}{2 \sin^2 \bar{x}_{k+1}}\right]$$

$$= \frac{4n}{\pi^2}\left(\frac{2}{(2k+1)^2} - \frac{1}{(2k+3)^2}\right) + o(n). \quad (33)$$

If $n^* = n-1$ $(x_m = \xi)$ then, holding $k < n/2$ for symmetry reasons we have

$$s_n^{*\prime}(\bar{x}_{m+k}) = \frac{1}{n}\left[(-1)^k \frac{\cos(\bar{x}_k)}{\sin^2(\bar{x}_k)} - 2 \sum_{j=2k+2}^{n/2+k} (-1)^{k+j} \frac{\cos(\bar{x}_{k-j})}{\sin^2(\bar{x}_{k-j})}\right].$$

Applying the lemma again we obtain as $n \to \infty$

$$(-1)^k s_n^{*\prime}(\bar{x}_{m+k}) > \frac{1}{n}\left[\frac{\cos(\bar{x}_k)}{\sin^2(\bar{x}_k)} - \frac{2\cos(\bar{x}_{k+1})}{\sin^2(\bar{x}_{k+1})} + \frac{\cos(\bar{x}_{k+2})}{\sin^2(\bar{x}_{k+2})}\right]$$

$$= \frac{4n}{\pi^2}\left[\frac{1}{(2k+1)^2} - \frac{2}{(2k+3)^2} + \frac{1}{(2k+5)^2}\right] + o(n) > 0, \quad (34)$$

$$(-1)^k s_n^{*\prime}(\bar{x}_{m+k}) < \frac{1}{n}\left[\frac{\cos(\bar{x}_k)}{\sin^2(\bar{x}_k)} - \frac{\cos(\bar{x}_{k+1})}{\sin^2(\bar{x}_{k+1})}\right]$$

$$= \frac{4n}{\pi^2}\left[\frac{1}{(2k+1)^2} - \frac{1}{(2k+3)^2}\right] + o(n). \quad (35)$$

Let $I_n =]x_m, x_m + \pi/2[$ if $x_m = \xi$ and $I_n =]\bar{x}_m, \bar{x}_m + \pi/2[$ if $x_m < \xi$. As a first conclusion of the foregoing estimates we see that in the nodes $x_{m+1}, x_{m+2}, ... \in I_n$ the function s_n^* is alternatingly increasing and decreasing (cf. (24), (28), (28′)), while in the points $\bar{x}_{m+1}, \bar{x}_{m+2}, ... \in I_n$ it is alternatingly decreasing and increasing (cf. (31), (32), (34)). The two sequences of bounds for the absolute values of the corresponding slopes are decreasing (cf. (26), (26′), (29), (29′) resp. (33), (35)). The intervals of length $\pi/2n$ to the right of the nodes $x_{m+1}, x_{m+2}, ... \in I_n$ contain alternatingly local maxima and local minima. This, together with the extremum in the right end point of I_n (cf. (27), (30)), accounts for $n/2 - 1$ zeros of $s_n^{*\prime\prime}$ in I_n, and in view of the inflection in the left end point of I_n, for $2n-2$ zeros of $s_n^{*\prime\prime}$ in $[-\pi, \pi[$. Any other local extreme of s_n^* in I_n would push the total of zeros of $s_n^{*\prime\prime}$ in $[-\pi, \pi[$ above $2n$; this is impossible for a trigonometric polynomial of order n. This establishes the following theorem.

THEOREM 2. *The function s_n^* assumes in each interval $[x_{m+k}, \bar{x}_{m+k}]$ $(1 \leqslant k \leqslant n/2 - 1)$ precisely one local extreme value, a maximum for $k = 2l - 1$ and a minimum for $k = 2l$. The derivative $s_n^{*\prime}$ has opposite signs in x_{m+k} and $\bar{x}_{m+k}$. The function s_n^* does not assume a local extreme value in $[x_m, x_{m+1}]$.*

4. Bounds

In order to obtain bounds for the extreme values of s_n^* we also try to clarify the convexity behaviour of s_n^* by evaluating the sign of $s_n^{*\prime\prime}$ in the points mentioned above.

In (18), again supposing $k \leqslant n/2$, the first $2k-2$ terms of the sum are pairwise equal while the remaining non-zero terms for $n^* = n$ cancel in pairs. We obtain

$$s_n^{*\prime\prime}(x_{m+1}) = -\frac{n^2-1}{3}$$

and for $k > 1$ and $n \to \infty$

$$\begin{aligned} s_n^{*\prime\prime}(x_{m+k}) &= 4 \sum_{j=1}^{k-1} (-1)^{j+1} \frac{\cos x_j}{\sin^2 x_j} - \frac{n^2-1}{3} \\ &= \frac{4n^2}{\pi^2} \left[\sum_{j=1}^{k-1} \frac{(-1)^{j+1}}{j^2} - \frac{\pi^2}{12} \right] + o(n^2). \end{aligned} \tag{36}$$

If $n^* = n-1$ then to the right member the term $(-1)^{k+1}\,(2 \cos{}_k/\sin^2 x^k)$ has to be added.

Observe that

$$\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j^2} = \frac{\pi^2}{12}$$

(this may be seen expanding the even periodic function given by $y = (x-\pi)^2$ on the period interval $[0, 2\pi]$ into a Fourier series). Therefore the difference in brackets is alternatingly positive (for even k) and negative (for odd k) and so is $s_n^{*\prime\prime}(x_{m+k})$ as $n \to \infty$, i.e.,

$$(-1)^k\, s_n^{*\prime\prime}(x_{m+k}) > 0 \qquad \text{for sufficiently large } n. \tag{37}$$

In case $n^* = n-1$ the main term of the last member has to be replaced by

$$\begin{aligned} &\frac{2n^2}{\pi^2} \left[\sum_{j=1}^{k-1} \frac{(-1)^{j+1}}{j^2} - \frac{\pi^2}{12} + \sum_{j=1}^{k} \frac{(-1)^{j+1}}{j^2} - \frac{\pi^2}{12} \right] \\ &= \frac{2n^2}{\pi^2} \left[\sum_{j=k}^{\infty} \frac{(-1)^j}{j^2} + \sum_{j=k+1}^{\infty} \frac{(-1)^j}{j^2} \right] \\ &= (-1)^k \frac{2n^2}{\pi^2} \sum_{j=0}^{\infty} \left[\frac{1}{(k+2j)^2} - \frac{2}{(k+2j+1)^2} + \frac{1}{(k+2j+2)^2} \right]. \end{aligned}$$

Since the series converges and its terms are positive the conclusion (37) about the sign of $s_n^{*\prime\prime}(x_{m+k})$ remains valid.

Finally, for $k \leqslant n/2$, in (19) the first $2k$ terms of the sum are pairwise equal while the remaining terms for $n^* = n$ cancel in pairs. This results in the formula

$$s_n^{*\prime\prime}(\bar{x}_{m+k}) = \frac{2}{n} \sum_{j=1}^{k} (-1)^j \frac{(n^2+1)\sin^2(\bar{x}_{j-1}) - 2}{\sin^3(\bar{x}_{j-1})} \tag{38}$$

$(n^* = n)$ where the term $(-1)^{k+1}((n^2+1)\sin^2(\bar{x}_k) - 2)/(n\sin^3(\bar{x}_k))$ has to be added if $n^* = n-1$. Observe that for fixed j and for $n \to \infty$ we have

$$a_j = \frac{n^2+1}{\sin(\bar{x}_{j-1})} - \frac{2}{\sin^3(\bar{x}_{j-1})} = n^3 \left[\frac{2}{\pi \cdot (2j-1)} - \frac{16}{\pi^3(2j-1)^3}\right] + o(n^3). \tag{39}$$

We therefore compare (38) with the corresponding partial sum of the series

$$\sum_{j=1}^{\infty} (-1)^j \left[\frac{2}{\pi \cdot (2j-1)} - \frac{16}{\pi^3(2j-1)^3}\right]. \tag{40}$$

Note again that

$$\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{2j-1} = \arctan 1 = \frac{\pi}{4}$$

and

$$\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{(2j-1)^3} = \frac{\pi^3}{32}$$

(this may be seen expanding the odd function y with period 8 and defined for $0 \leqslant x \leqslant 4$ by $y = 4x - x^2$ into a Fourier series). Therefore the series (40) converges to zero. The function

$$g(x) = \frac{1}{x} - \frac{8}{\pi^2 x^3}$$

is positive for $x > 2\sqrt{2}/\pi = 0{,}900\ldots$, decreasing for $x > 2\sqrt{6}/\pi = 1{,}559\ldots$, and convex for $x > 4\sqrt{3}/\pi = 2{,}205\ldots$ (this will allow an application of Lemma 1 below). Also, one has $g(1) = 1 - 8/\pi^2 = 0{,}189\ldots$ and $g(3) = 1/3 - 8/(27\pi^2) = 0{,}303\ldots$. Therefore in the series (40) not only the individual terms but also the partial sums are alternating in sign. By (39) the same is true for (38) as $n \to \infty$, i.e. for fixed k and for sufficiently large n we have

$$(-1)^k s_n^{*\prime\prime}(\bar{x}_{m+k}) > 0. \tag{41}$$

In order to verify the same assertion for $n^* = n-1$ we observe that

$$\begin{aligned}(-1)^k \sum_{j=1}^{k} (-1)^j (2j-1) \\ &= \sum_{j=k+1}^{\infty} (-1)^{j+k+1} g(2j-1) \\ &= [g(2k+1) - g(2k+3)] + [g(2k+5) - g(2k+7)] + \cdots.\end{aligned}$$

By the convexity of the function g we may apply a reasoning as in the proof of the lemma and obtain

$$(-1)^k \sum_{j=1}^{k} (-1)^j g(2j-1) > (-1)^{k+1} \sum_{j=1}^{k+1} (-1)^j g(2j-1).$$

As $n \to \infty$ we get the same inequalities for the partial sums of the series $\sum_{j=1}^{\infty} (-1)^j a_j$. Now for $n^* = n-1$ we have

$$(-1)^k s_n^{*\prime\prime}(\bar{x}_{m+k}) = \frac{(-1)^k}{n} \sum_{j=1}^{k} (-1)^j a_j + \frac{(-1)^k}{n} \sum_{j=1}^{k+1} (-1)^j a_j > 0. \tag{42}$$

Thus by (37) and (41) for each $l \in \mathbb{N}$ the function s_n^* is eventually concave in x_{m+2l-1} and $\bar{x}_{m+2l-1}$ and convex in x_{m+2l} and $\bar{x}_{m+2l}$. In the reasoning preceding Theorem 2 we have seen that $s_n^{*\prime\prime}$ can change sing between consecutive extreme values only once. Therefore, for fixed $p \in \mathbb{N}$ and $1 \leqslant k \leqslant p$ the corresponding zeros of $s_n^{*\prime\prime}$ eventually have to lie in the intervals $]\bar{x}_{m+k}, x_{m+k+1}[$ and the function s_n^* eventually has to be concave in $[x_{m+2l-1}, \bar{x}_{m+2l-1}]$ and convex in $[x_{m+2l}, \bar{x}_{m+2l}]$.

THEOREM 3. *Given any $k \in \mathbb{N}$, for sufficiently large n the values*

$$s_n{}^*(\bar{x}_{m+k}) - \frac{\pi}{2n} s_n^{*\prime}(\bar{x}_{m+k}) \tag{43}$$

(as well as the values $s_n^*(x_{m+k}) + (\pi/2n)\, s_n^{*\prime}(x_{m+k})$) furnish upper resp. lower bounds for the corresponding maxima resp. minima of s_n^* in the intervals $[x_{m+k}, \bar{x}_{m+k}]$.

A comparison of the values of $s_n^*(\bar{x}_{m+l})$ as calculated in (11), (12) with (43) (taking into account the bounds (33) and (35)) shows that as $n \to \infty$ the largest overshot occurs in the first interval $]x_{m+1}, \bar{x}_{m+1}]$.

Since s_n^* as defined in (7) has eventually to be concave in the entire interval $[x_{m+2l-1}, \bar{x}_{m+2l-1}]$ and convex in the entire interval $[x_{m+2l}, \bar{x}_{m+2l}]$ an even better upper resp. lower estimate for the maxima resp. minima may be obtained by the ordinate y_0 of the point of intersection of the two tangents in the endpoints. Let the ordinates in these endpoints be denoted by y_1 resp. y_2 and the corresponding slopes by k_1 resp. k_2. Then with $h=\pi/2n$ one has

$$y_0=\frac{k_1 y_2-k_2 y_1-k_1 k_2 h}{k_1-k_2}.$$

Observe now that the same convexity property of s_n^* in the interval $[x_{m+k}, \bar{x}_{m+k}]$ together with the information on the inclinaton of the tangents in x_{m+k} and $\bar{x}_{m+k}$ contained in (24), (28), (28′), (31), (32), (34) allows us to estimate the maximum from above resp. the minimum from below if the tangents are replaced by "steeper" straight lines, i.e., if the slopes of the tangents are replaced by upper estimates of positive values and by lower estimates of negative values, as furnished by (26), (26′), (29), (29′) for k_1 resp. by (33), (35) for k_2. Applying this to the interval $[x_{m+1}, \bar{x}_{m+1}]$ in the case

$$x_m=\xi \qquad \text{resp.} \quad x_m<\xi$$

with

$$y=1$$

$$y_2=\frac{10}{3\pi}+o(1) \qquad \text{resp.} \quad y_2=\frac{4}{\pi}+o(1)$$

$$k_1=\frac{n}{2\pi}+o(n) \qquad \text{resp.} \quad k_1=\frac{3n}{2\pi}+o(n)$$

$$k_2=-\frac{4n}{\pi^2}\left(\frac{1}{9}-\frac{1}{25}\right)+o(n) \qquad \text{resp.} \quad k_2=-\frac{4n}{\pi^2}\left(\frac{2}{9}-\frac{1}{25}\right)+o(n)$$

as $n\to\infty$ we obtain

$$y_0=1{,}090\ldots+o(1) \qquad \text{resp.} \quad y_0=1{,}337\ldots+o(1)$$

as an upper bound for the corresponding maximum. This maximum in turn should provide the asymptotic overshot factor $\bar{S}$ in the formula below. Numerical evidence seems to indicate that under the suppositions of the theorem

$$\sup_{x_{m+1}\leqslant x\leqslant \bar{x}_{m+1}} s_n^*(x)-\frac{f^+(\xi)+f^-(\xi)}{2}=s\cdot\bar{S}(\xi-x_m)+o(1),$$

where

$$\bar{S}(0)\approx 1{,}065\ldots$$

$$\bar{S}(t)\approx 1{,}282\ldots \qquad \text{for} \quad t>0.$$

The concavity of s_n^* in $\bar{x}_{m+2l-1}$ (and similarly the convexity of s_n^* in $\bar{x}_{m+2l}$) could also have been established using the following geometric argument: the tangent to the graph of s_n^* at $\bar{x}_{m+2l-1}$ intersects the line $y=1$ to the right of x_{m+2l}. This again can be shown using the corollary and the upper estimate for $|s_n^{*\prime}(\bar{x}_{m+2l-1})|$.

5. Odd Half Number of Nodes

The case of an odd n does not differ essentially from the case of an even n. It is governed by the formulas

$$s_{nn}^*(x)=\frac{(-1)^m}{n}\sum_{j=1}^{n^*}(-1)^{j+1}\sin nx\cot(x_{m+j}-x)$$

$$s_n^{*\prime}(x)=\frac{(-1)^m}{n}\sum_{j=1}^{n^*}(-1)^{j+1}\times\frac{n\cos nx\cos(x_{m+j}-x)\sin(x_{m+j}-x)+\sin nx}{\sin^2(x_{m+j}-x)}$$

$$s_n^{*\prime\prime}(x)=\frac{(-1)^m}{n}\sum_{j=1}^{n^*}(-1)^{j+1}\left\{\frac{2n\cos nx\sin(x_{m+j}-x)}{\sin^3(x_{m+j}-x)}+\frac{\sin nx\cos(x_{m+j}-x)[2-n^2\sin^2(x_{m+j}-x)]}{\sin^3(x_{m+j}-x)}\right\}$$

$$s_n^{*\prime}(x_{m+k})=\sum_{\substack{j=1\\ j\neq k}}^{n^*}(-1)^{k+j+1}\cot(x_{j-k})$$

$$s_n^{*\prime}(\bar{x}_{m+k})=\frac{1}{n}\sum_{j=1}^{n^*}\frac{(-1)^{j+k+1}}{\sin^2(\bar{x}_{j-k-1})}$$

$$s_n^{*\prime\prime}(x_{m+k})=\sum_{\substack{j=1\\ j\neq k}}^{n^*}(-1)^{k+j+1}\frac{2}{\sin^2(x_{j-k})}-\frac{n^2+2}{3}$$

$$s_n^{*\prime\prime}(\bar{x}_{m+k})=\frac{1}{n}\sum_{j=1}^{n^*}(-1)^{k+j+1}\cos(\bar{x}_{j-k-1})\frac{2-n^2\sin^2(\bar{x}_{j-k-1})}{\sin^3(\bar{x}_{j-k-1})}.$$

These formulas provide similar finite and the same asymptotic estimates for $n\to\infty$ as in case of an even n. As to the number of extreme values of s_n^*, however, there is a difference to be noted in comparison with the case of an even n: For $x_m<\xi$ each interval $[x_{m+j}, x_{m+j+1}[$ $(1\leqslant j\leqslant(n-1)/2)$ contains an extreme value. Together with the extreme value in $x_{m+(n+1)/2}=\bar{x}_m+\pi/2$, a symmetry point for s^*, this accounts for n extreme values in $[\bar{x}_m, \bar{x}_m+\pi[$, as much as a trigonometric polynomial of order n can have, considering the symmetry of s_n^* with respect to $\bar{x}_m$. For $x_m=\xi$, however, there are extreme values in $[x_{m+j}, x_{m+j+1}[$ $(1 \leqslant j \leqslant (n-3)/2)$ plus one at the symmetry point $\bar{x}_{m+(n-1)/2}=x_m+\pi/2$ of s_n^*. This accounts for $n-2$ extreme values in $[x_m, x_m+\pi[$. The followig lemma should forestall a useless search for supposedly forgotten extreme values.

LEMMA 2. *For f as in (6) the order of s_n^* as a trigonometric polynomial is*

(a) $n-1$ *for n even,*
(b) n *for n odd and $x_m<\xi$,*
(c) $n-2$ *for n odd and $x_m=\xi$.*

Proof. Having exhibited the corresponding numbers of extreme values in a symmetry-half of a period interval it suffices to show that the order is bounded above by the given values. Therefore (b) is evident. Now consider

$$ns_n^*(x)=\sum_{j=1}^{n^*}\sum_{k=1}^{n-1}[\cos k(x-x_j)-\cos k(x-x_j-\pi)]$$
$$+\frac{1}{2}\sum_{j=1}^{n^*}[\cos n(x-x_j)-\cos n(x-x_j-\pi)].$$

The difference between the last brackets vanishes if n is even. This shows (a). If n is odd and $x_m=\xi$ then $n^*=n-1$ is even and the last sum becomes

$$\sum_{j=1}^{n-1}\cos(nx-j\pi)=\cos nx\cdot\sum_{j=1}^{n-1}(-1)^j=0.$$

Moreover, in the first sum for $k=n-1$ the difference between the brackets vanishes since $n-1$ is even. This shows (c). ∎

As an example we note that for $n=3$ and $\xi=0$ one gets

$$s_3^*(x)=\frac{2}{\sqrt{3}}\sin x.$$

The further conclusions about maxima and minima of s_n^* are the same as in the case of an even n. The details are left to the reader.

6. ODD NUMBER OF NODES

So far (cf. (1)) we have considered interpolation with an even number of nodes $x_j=x_{n,j}=j\pi/n$ $(1 \leqslant j \leqslant 2n)$ in the period interval $]0, 2\pi]$. There are several excuses for doing so. In the first place, due to the fact that together with $x_j + \pi = x_{j+n}$ is a node, the function s_n^* is given by comparatively simple formulas. Secondly, an analysis of this function is simplified by various symmetries which decompose s_n^* into four congruent parts within an appropriate period interval. Finally, the usual fast Fourier transform is based on powers of two as a number of nodes. Essentially, however, not much changes if we use an odd number of equidistant nodes

$$x_j = x_{n,j} = 2j\pi/(2n+1) \qquad (-n \leqslant j \leqslant n) \tag{44}$$

in a period interval $]-\pi, \pi]$ (we freely admit nodes which are congruent to the above ones modulo 2π). Since it does not seem worth while to go through all details again we sketch an approach by which the reader can convince himself of this fact, using a rearrangement of the calculations which also sheds some more light on the possibility of cancelling and doubling which has been helpful in the previous calculations.

In general for a function f the unique trogonometric polynomial $\bar{s}_n$ of order $\leqslant n$ interpolating f at the nodes (44) is now given by

$$\bar{s}_n(x) = \frac{2}{2n+1} \sum_{j=-n}^{n} f(x_j)\, D_n(x - x_j),$$

where the Dirichlet kernel D_n is given by

$$D_n(u) = \frac{1}{2} + \sum_{k=1}^{n} \cos ku = \frac{\sin(((2n+1)/2)u)}{2\sin(u/2)}$$

[9, X.1]. D_n is again an even function satisfying

$$D_n(\pi - x) = D_n(\pi + x)$$

$$D_n(0) = n + \frac{1}{2},$$

$$D_n(x_j) = 0 \qquad \text{for} \quad j \not\equiv 0 \mod 2n+1,$$

$$D_n(\pi) = \frac{(-1)^n}{2},$$

$$D_n'(u) = -\sum_{k=1}^{n} k \sin ku$$

$$= \frac{(2n+1)\cos(((2n+1)/2)u)\sin(u/2) - \sin(((2n+1)/2)u)\cos(u/2)}{4\sin^2(u/2)}$$

$$D_n'(0) = D_n'(\pi) = 0,$$

$$D_n''(u) = -\sum_{k=1}^{n} k^2 \cos ku$$
$$= \frac{\begin{bmatrix} -(2n+1)\cos(((2n+1)/2)u)\sin u \\ \qquad -2\sin(((2n+1)/2)u)[2n(n+1)\sin^2(u/2)-\cos^2(u/2)] \end{bmatrix}}{8\sin^3(u/2)},$$

$$D_n''(0) = -\sum_{k=1}^{n} k^2 = -\frac{n}{6}(n+1)(2n+1),$$

$$D_n''(\pi) = \sum_{k=1}^{n} (-1)^{k-1} k^2 = (-1)^{n+1}\frac{n}{2}(n+1).$$

The functions D_n' resp. D_n'' are odd resp. even.

In the sequel also the following values will be needed (we adhere to the notation (15) but now referring to the nodes (44))

$$D_n(\bar{x}_j) = \frac{(-1)^j}{2\sin(\bar{x}_j/2)},$$

$$D_n'(x_j) = \frac{(-1)^j(2n+1)}{4\sin(x_j/2)},$$

$$D_n'(\bar{x}_j) = (\frac{-1)^{j+1}\cos(\bar{x}_j/2)}{4\sin^2(\bar{x}_j/2)},$$

$$D_n''(x_j) = \frac{(-1)^{j+1}(2n+1)\cos(x_j/2)}{4\sin^2(x_j/2)}$$

$$D_n''(\bar{x}_j) = (-1)^{j+1}\left[\frac{n(n+1)}{2\sin(\bar{x}_j/2)} - \frac{\cos^2(\bar{x}_j/2)}{4\sin^3(\bar{x}_j/2)}\right].$$

For the study of Gibbs' phenomenon again as in the proof of the theorem and in the sequel thereof it suffices to consider the function f given by (6). Without loss of generality we now assume $-\pi/(2n+1) < \xi \leqslant \pi/(2n+1)$. The interpolating polynomial $\bar{s}_n$ is then given by

$$\begin{array}{ll}
\dfrac{2}{2n+1}\displaystyle\sum_{j=1}^{n} [D_n(x-x_j) - D_n(x+x_j)] & \text{if } \xi = 0, \\
\dfrac{2}{2n+1}\left\{\displaystyle\sum_{j=1}^{n} [D_n(x-x_j) - D_n(x+x_j)] - D_n(x)\right\} & \text{if } 0 < \xi < \dfrac{\pi}{2n+1}, \\
\dfrac{2}{2n+1}\left\{\displaystyle\sum_{j=1}^{n} [D_n(x-x_j) - D_n(x+x_j)] + D_n(x)\right\} & \text{if } -\dfrac{\pi}{2n+1} < \xi < 0, \\
\dfrac{2}{2n+1}\displaystyle\sum_{j=1}^{n} [D_n(x-x_j) - D_n(x+x_{j-1})] & \text{if } \xi = \dfrac{\pi}{2n+1}.
\end{array}$$

A glance at the case $\xi = 0$ will exhibit the computational effect of the underlying symmetries: if we choose $x = x_k$ or $x = \bar{x}_k$ then, as demonstrated below, the terms which appear in both of the sums

$$\sum_{j=1}^{n} D_n(x - x_j) - \sum_{j=1}^{n} D_n(x + x_j)$$

cancel while the remaining terms pair off due to the fact that D_n is even. This works the same way for the second derivatives, while in the first derivative of $\bar{s}_n$ the effect is different due to the fact that D_n' is odd. In contrast to the interpolated function f, its interpolating polynomial $\bar{s}_n$ is no longer symmetric with respect to $x = \pi/2$. Still we mainly restrict attention to $0 \leqslant k \leqslant n/2$.

For $n \to \infty$ we obtain

$$\begin{aligned}
\bar{s}_n(\bar{x}_k) &= \frac{2}{2n+1} \sum_{j=1}^{n} [D_n(\bar{x}_{k-j}) - D_n(\bar{x}_{k+j})] \\
&= \frac{2}{2n+1} \left[\sum_{j=0}^{k-1} \frac{(-1)^j}{\sin(\bar{x}_j/2)} + \frac{(-1)^k}{2\sin(\bar{x}_k/2)} - \sum_{j=n-k}^{n-1} \frac{(-1)^j}{\sin(\bar{x}_j/2)} - \frac{(-1)^n}{2} \right] \\
&= \frac{2}{\pi} \left[2 \sum_{j=0}^{k-1} \frac{(-1)^j}{2j+1} + \frac{(-1)^k}{2k+1} \right] + o(1).
\end{aligned}$$

This agrees with the assertions (11), (12) for the case $x_m = \xi$ in the corollary. In particular for $k = 1$ we get

$$\begin{aligned}
\bar{s}_n\left(\frac{3\pi}{2n+1}\right) &= \frac{2}{\pi}\left[2 - \frac{1}{3}\right] + o(1) \\
&= \frac{10}{3\pi} + o(1),
\end{aligned}$$

while a similar calculation gives

$$\begin{aligned}
\bar{s}_n\left(\frac{2n-1}{2n+1}\pi\right) &= \frac{2}{2n+1}\left[\frac{1}{\sin(\bar{x}_0/2)} - \frac{(-1)^{n-1}}{2\sin(\bar{x}_{n-1}/2)} - \frac{(-1)^n}{2}\right] \\
&= \frac{4}{\pi} + o(1),
\end{aligned}$$

both in agreement with Theorem 1(5).

Similarly we have

$$\begin{aligned}
\bar{s}_n'(x_k) &= \frac{2}{2n+1} \sum_{j=1}^{n} [D_n'(x_{k-j}) - D_n'(x_{k+j})] \\
&= \sum_{j=k+1}^{n-k} \frac{(-1)^{j+1}}{\sin(x_j/2)} + \frac{(-1)^{k+1}}{2\sin(x_k/2)}.
\end{aligned}$$

Applying Lemma 1 and letting $n \to \infty$ we get the same asymptotic estimates (for $m=0$) as in (28), (29), (34), (35), (36), (42) (the reader interested in details is referred to [3]).

The conclusion is that in the points x_k and $\bar{x}_k$ the interpolating trigonometric polynomial behaves asymptotically in the same way for an odd number of nodes as for an even number of nodes. It would be convenient to obtain this scarcely surprising conclusion and its extension to the whole period interval by a uniform $o(1)$-estimate of $|s_n^* - \bar{s}_n|$ as $n \to \infty$. Unfortunately at least over the entire period interval such an estimate seems impossible since

$$\bar{s}_n\left(\frac{2n}{2n+1}\pi\right) = 1$$

while

$$s_n^*\left(\frac{2n-1}{2n}\pi\right) = \frac{1}{2n}\left[\cot(\pi/4n) + (-1)^n \cot((2n-1)\pi/4n)\right] = \frac{2}{\pi} + o(1)$$

and

$$\frac{2n-1}{2n}\pi < \frac{2n}{2n+1}\pi < \pi.$$

7. Asymptotic Behaviour

If a sequence of interpolations is considered where n runs through an increasing sequence of integers $\{n_i\}_{i=1}^{\infty}$ then the behaviour of s_n^* and $\bar{s}_n$ in the neighbourhood of an isolated jump discontinuity ξ of f depends on the position of ξ with respect to the nodes considered. If $\xi = 0$ then the overshot is governed by the case $x_m = \xi$. The same is true if ξ is a node for every n_i $(1 \leqslant i < \infty)$, e.g., if ξ is a dyadic fraction of 2π as in the usual application of the fast Fourier transform. If ξ is an irrational multiple of 2π then the overshot is governed by the case $x_m < \xi$. If ξ is non-zero and a rational multiple of 2π and if the number of nodes runs through $\mathbb{N}$, then infinitely often the overshot behaviour changes from one case to the other. In particular for the function (6) this alternating behaviour may be observed at the point $\bar{\xi} = \pi$ if $\xi = 0$ and if the number of nodes increases through the sequence of natural numbers. In any case, the asymptotic overshot factors differ from the factor turning up in the classical Gibbs phenomenon, as shown by the calculated lower and upper bounds.

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